Source Unfolding

Joseph Thomas

University of Arizona
Department of Mathematics

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Polytopes and their Boundaries

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- $S := \partial P$.

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Questions:

- How to visualize and understand S?
- How can one deal with S computationally?

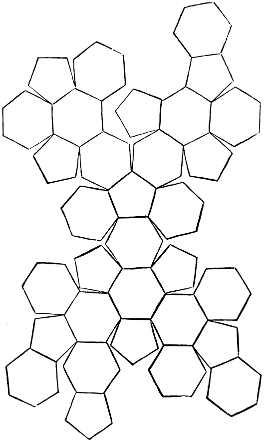
Overview

Dürer's Conjecture

An open problem from the 1500's:

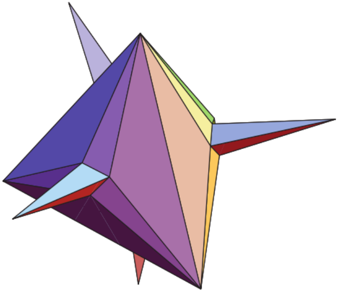
Conjecture

Suppose $P \subset \mathbb{R}^3$ is a polytope. Then ∂P can be cut along its edges and unfolded into a single connected subset of the plane.



One of Dürer's unfoldings.

Dürer's Conjecture



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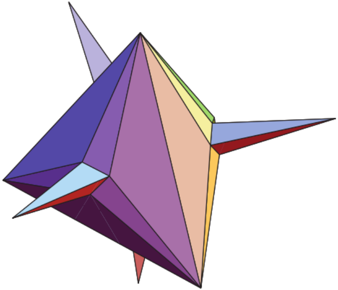

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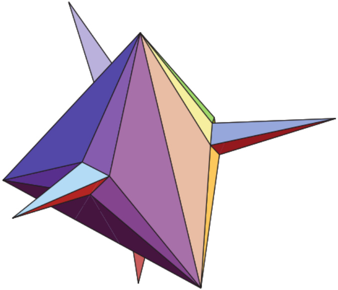

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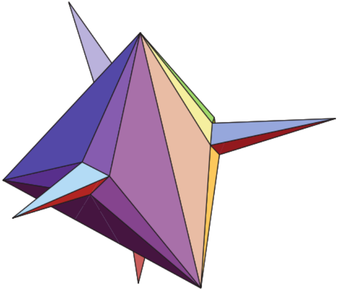

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- Convexity is necessary (see left).
- Combinatorics: How are edges, faces, and vertices connected?
- Geometry: How do lengths, angles, and areas affect the unfolding?

Source Unfolding

Fix a **source point** $v \in S$. A **source unfolding** $\varphi : S \setminus K \to \mathbb{R}^d$ is a map such that:

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- Cuts are allowed to "slice" through faces of S.
- In our construction, shortest paths starting at v unfold to straight lines.

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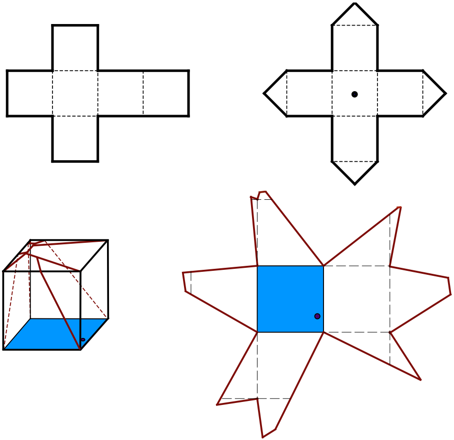
- Optimization — φ records shortest paths from the source point to any other point in S (similar to Dijkstra's algorithm).

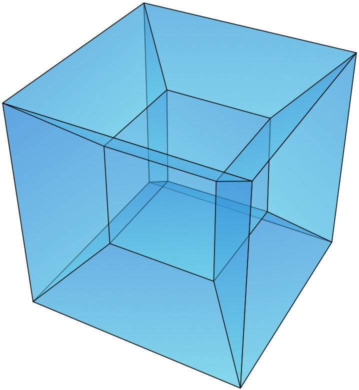
Example: A Cube

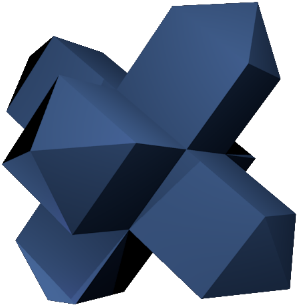


Edge and source unfoldings of a cube.

Image Credit: Ezra Miller and Igor Pak

Example: A Hypercube





A “stereographic projection” of a hypercube

and a source unfolding.

Image Credit: John Baez/Creative Commons (left)

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Q: What "standard" tools are available for understanding polytopes?

A: Linear algebra, lattice theory, and techniques from linear programming.

Visualizing Polytopes

Visualizing Polytopes

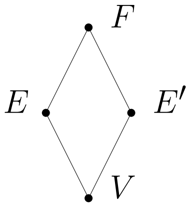
Here, however, a word of warning may be in order: do **not** *try to visualize n-dimensional objects for* $n \geq 4$*. Such an effort is not only doomed to failure—it may be dangerous to your mental health. (If you do succeed, then you are in trouble.)*

—V. Chvátal

Polytope Combinatorics

Proposition (Diamond Property for Face Containments)

Suppose $F \supset V$ are k and $(k-2)$-dimensional faces respectively. There exist exactly two $(k-1)$-dimensional faces E, E' so that $V \subseteq E, E' \subseteq F$.



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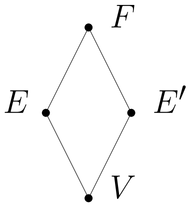

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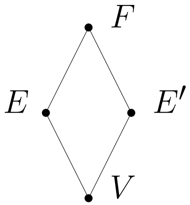

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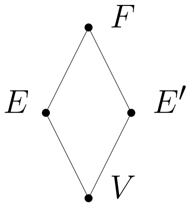

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- When two faces meet, they do so along exactly one edge.

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- $\sum\{\text{face angles about } V\} < 2\pi$.
- If $x \in \operatorname{relint} V$, there is a neighborhood of x isometric to a neighborhood of $\mathbb{R}^{d-2} \times C$, C a polyhedral cone (dimension 2).

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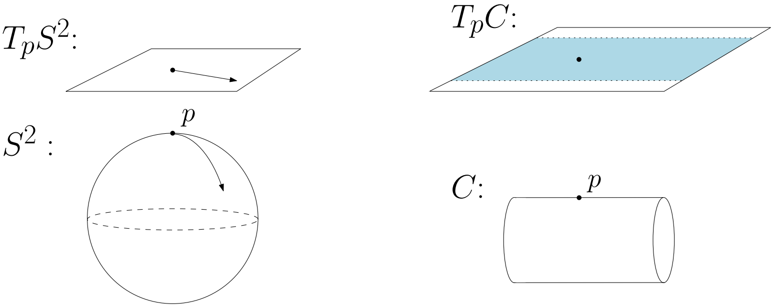
(iii) If you can't do (ii), then $v \notin \mathcal{E}$.

Riemannian Geometry — The Exponential Map



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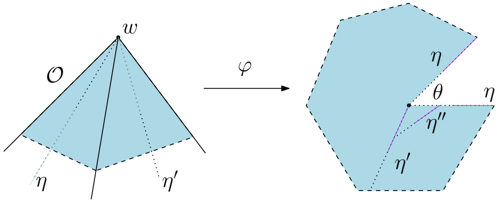
Conclusion: A shortest paths γ is specified by its endpoints and the sequence of d-faces it traverses.

Geodesics avoid k-faces, $k \leq d - 2$

Suppose γ is a geodesic ($\gamma = \eta.\eta'$) passing through w in a k-face F.



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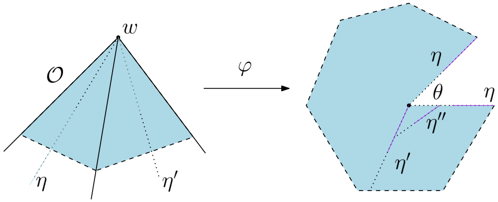

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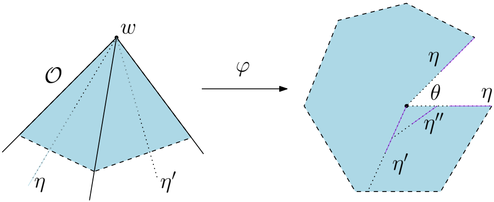

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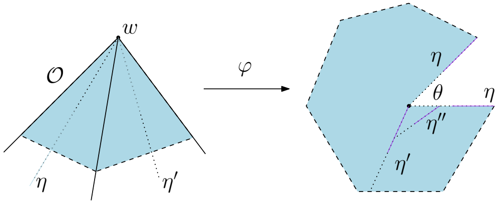

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$d > 2$ η, η' determine a plane that intersects F at a single point, w. Project onto this plane to reduce to case $d = 2$.

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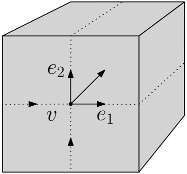

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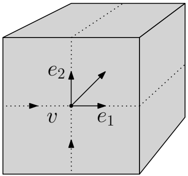

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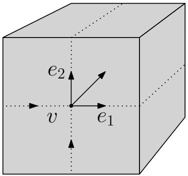

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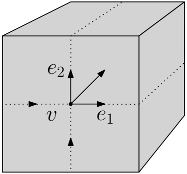

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- Not every vector can be exponentiated.

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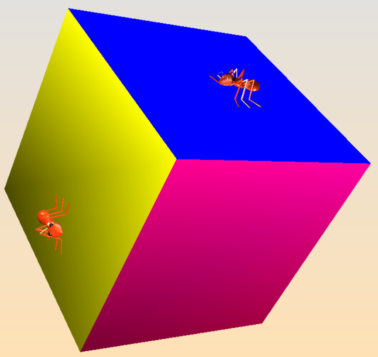

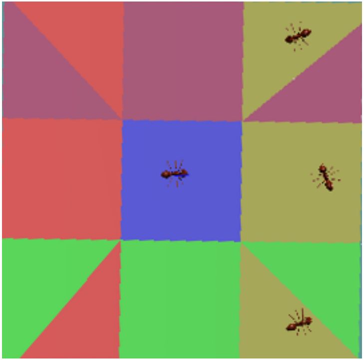

Image Credit : Dave Glickenstein's GEOCAM project (of which I am a member).

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- We want to use these maps to understand exp.
- We unfold a shortest path $v \to w \in F$ to get a point $\nu \in T_F$.

Exponential Maps for Polytopes

Given adjacent d-faces F, F', we have a unique isometry that rotates T_F into $T_{F'}$.

- We can compose these maps and invert to unfold paths.
- Different compositions might "overlap."
- We want to use these maps to understand exp.
- We unfold a shortest path $v \to w \in F$ to get a point $\nu \in T_F$.
- Do this for every $w \in F$, to obtain a set of **source images** src_F.

Source Images

Problem: Where should we make cuts to use $\exp^{-1}$ to unfold S?

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- Classify points in F by their nearest source image.
- To Be Shown: If $w \in F$ is nearest to $\nu \in \mathrm{src}_F$, this has implications for shortest paths $v \to w$.

Voronoi Diagrams

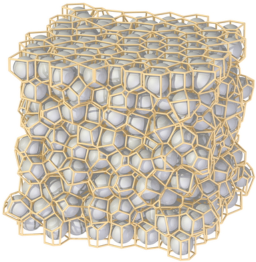

Image Credit: Chris H. Rycroft

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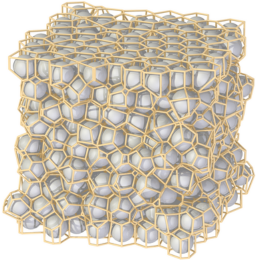

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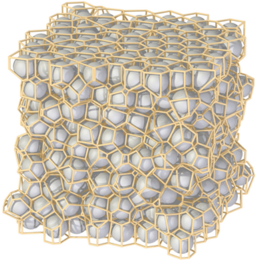

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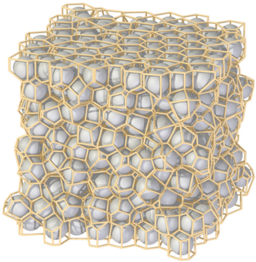

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- The **Voronoi diagram** determined by U is a cover of $\mathbb{R}^n$ by closed **Voronoi cells** $V(u, U)$.
- $V(u, U) = \{x \in \mathbb{R}^n : d(x, u) \leq d(x, u') \, \forall u' \in U\}$

Mount's Lemma

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A: The source image ν might not have unfolded from a shortest path $v \to w$.

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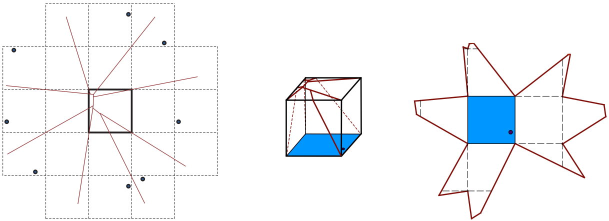

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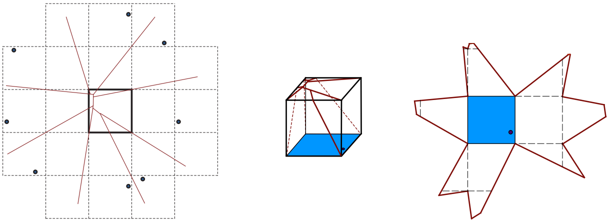

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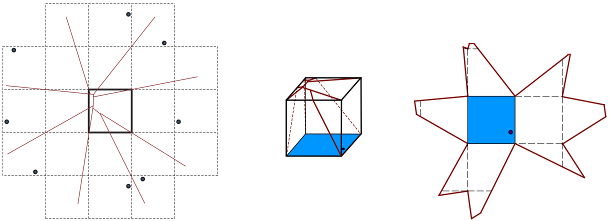

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Q: What does Mount's Lemma say about the Voronoi cells formed by src_F?

A: Points in $F \cap V(\nu, \text{src}_F)$ are **precisely** the points p so that a shortest path $v \to p$ unfolds to $[p, \nu]$!

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- V tells us how to "cut" S for a source unfolding!
- V is the closure of the **cut locus** (the set of points with multiple shortest paths to v).
- The algorithmic situation:

$$\text{input } P \rightarrow \text{src}_F \rightarrow V \rightarrow \{\text{cut points}\} \rightarrow \text{an unfolding}$$

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Idea #2: It suffices to prove the following:

If no shortest path $v \to w$ unfolds to $[\nu, w]$, then w is strictly closer to some other source image ν'.

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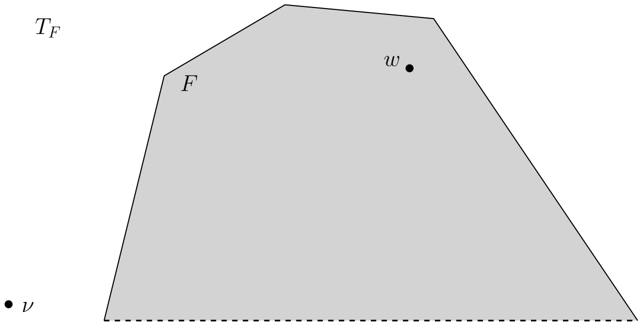

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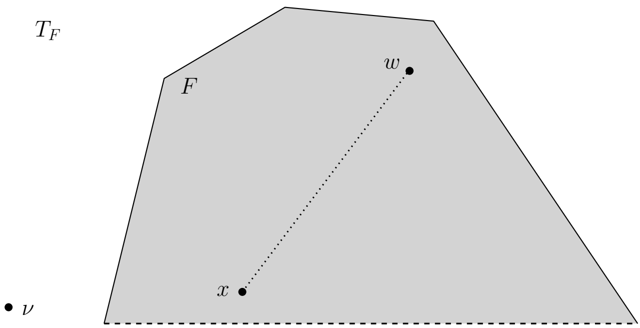

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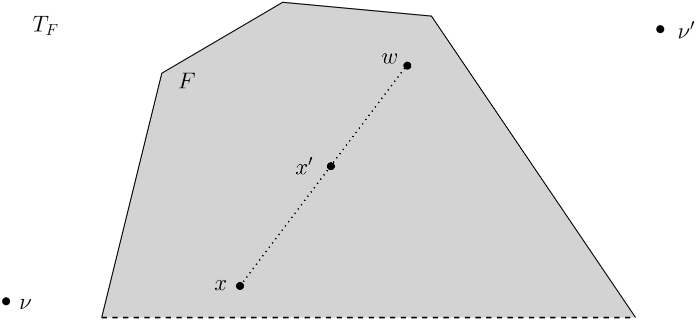

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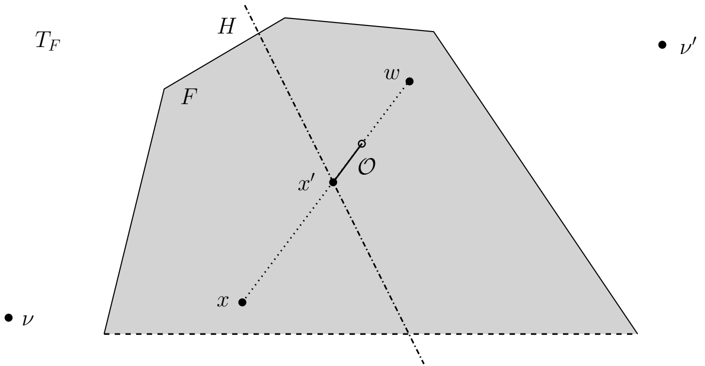

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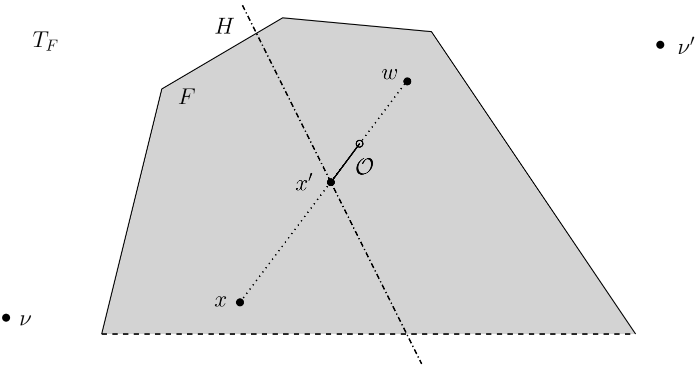

$$\forall y \in \mathcal{O} \setminus x' : L[\nu, y] > \mu(y, v)$$

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- So γ is actually a shortest path.

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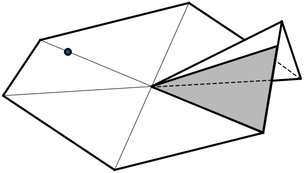
With these results gathered, it isn't hard to show $\exp^{-1} : S \setminus C_v \to T_v S$ is an unfolding map.

Further Questions



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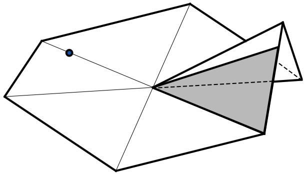

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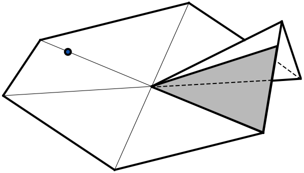

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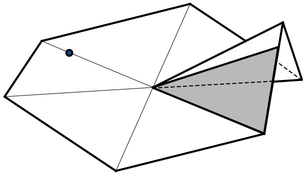


- How can one deal with a non-convex polytope P?
- How can one compute the source images?
- Can one characterize the number of source images (asymptotically)?

References

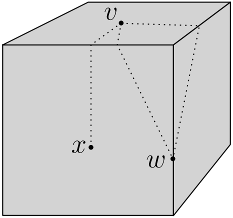
Miller, Ezra and Igor Pak. *Metric combinatorics of convex polyhedra: cut loci and nonoverlapping unfoldings*, Discrete and Computational Geometry 39 (2008), no. 1-3, 339-388.

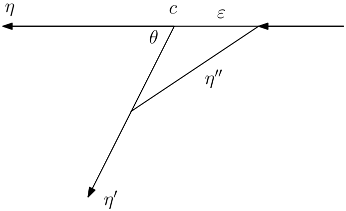
Ziegler, Günter M. *Lectures on Polytopes*, Graduate Texts in Mathematics 152, Springer-Verlag New York 1995, Revised sixth printing 2006.

Thanks!

Shortest paths from v avoid cut points

A **cut point** is a point with multiple shortest paths to v.





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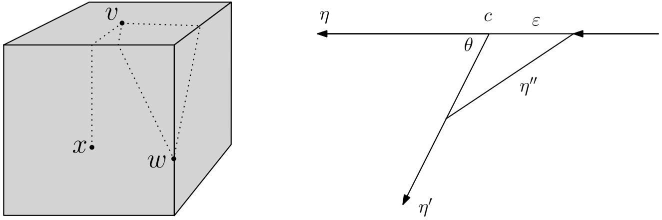
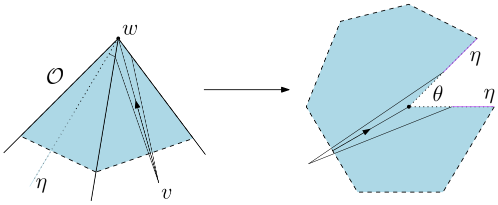


A path through a cut point c can be shortened by choosing one of the shortest paths $v \to c$ and then "cutting the corner."

All $(d-2)$-faces are contained in the cut locus.

Plan: Show each point in a $(d-2)$-face F is a limit point of C_v.
When $d=2$:



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- Use this characterization to find cut points near w.